PH205:Mathematical Methods of Physics

Date: 09-12-2016 End term Duration: 10 AM to 1 PM

[Total Marks:100. Q1=10, Q2 = 10, Q3=15, Q4=20, Q5=15, Q6 = 30]

We have learned four very useful theorems related to vector integrals

- Green's theorem for line integral

$$\int_a^b \frac{\partial f(s)}{\partial s} = f(b) - f(a).$$

- Green's theorem in a plane

$$\int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C (P\, dx + Q\, dy),$$

  where S is the area enclosed by the boundary C.

- Stoke's theorem

$$\int\int_S \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right).d\vec{s} = \oint_C \overrightarrow{A}.d\vec{r}.$$

- Gauss' law

$$\int_V \left(\overrightarrow{\nabla}.\overrightarrow{A}\right) dV = \oint_S \overrightarrow{A}.d\overrightarrow{S},$$

  where S is a closed area encircling the total volume V.

Q1: Answer either (a) or (b)

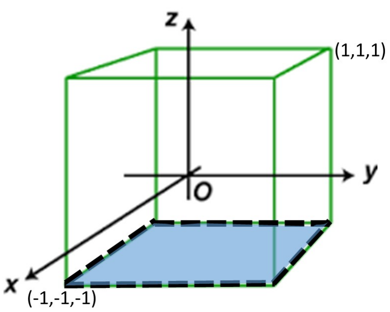
(a) (i) Using Green's theorem, prove Stoke's theorem. (4)

(ii) For $\overrightarrow{A}$ is any vector, prove

$$\oint_S d\overrightarrow{S} \times \overrightarrow{A} = \int_V \overrightarrow{\nabla} \times \overrightarrow{A} dV$$

(6)

(b) Using Stokes theorem evaluate the flux of a magnetic field $\int\int_S \overrightarrow{B}.d\overrightarrow{S}$ where $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$ and the vector potential is $\overrightarrow{A} = xyz\hat{x} + xy\hat{y} + x^2yz\hat{z}$ and let S be the the top and all four sides **but not the bottom** side of a cube with vertices, $(\pm1, \pm1, \pm1)$. [See attached figure where the bottom side is empty]



(10)

Q2: Solve either (a) or (b)

(a) Determine whether the following matrix A is diagonalizable.

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix},$$

(3)

If yes, find the corresponding diagonal matrix D and the matrix U that diagonalizes A as $U^{-1}AU = D$. Also prove that U is an unitary matrix. (6+1)

(b) Dirac or Gamma matrices are defined as $\Gamma_{\mu\nu} = \sigma_\mu \otimes \sigma_\nu$ where $\otimes$ is the Direct product and $\sigma_\nu(\nu = 0, x, y, z)$ are $2 times 2$ identity matrix and Pauli matrices defined as
$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\boldsymbol{I}_{4\times4}$ is 4×4 identity matrix.
Then prove that

$$e^{i\Gamma_{\mu\nu}\theta} = \boldsymbol{I}_{4\times4}\ cos\theta + i\Gamma_{\mu\nu}\ sin\theta$$

for any real θ. (10)

[Useful identities:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1},$$
$$\cos A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n},$$
$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$]

Q3: Answer either (a) or (b)

(a) Use a rectangular contour or any other contour of your convenience to prove that

$$\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{sin(\pi n)},$$

for $0 < n < 1$.

[Marks distributions: Total 15: Finding branch points/poles/branch cut (2 mark), Drawing contour (1 mark), Integral formula and Residue theorem (2 mark) and Solution (10 marks)]

(b) Show that

$$\int_0^1 \frac{1}{(x^2 - x^3)^{\frac{1}{3}}}\ dx = \frac{2\pi}{\sqrt{3}},$$

[Marks distributions: Total 15: Finding branch points and branch cut (2 mark), Drawing contour (1 mark), Integral formula and Residue theorem (2 mark), Solution (10 marks)]

Q4: Solve any <u>two</u>: We want to expand $f(z) = \sin(az)$ in any two following methods. (10+10)

(a). Laurent/Taylor series when $a > 0$ and z is a complex number. Use the contour integration formula derived in the class to evaluate the coefficients. If you can show that the contour integrals can be converted into complex derivative of the function $f(z)$, only then derivatives can be used to solve for the coefficients. [Expand upto z^3 term.] (9)

Evaluate the circle of convergence. (1)

(b). Expand the same function $f(z) = \sin(az)$ where z is real, in terms of the Legendre polynomials $P_n(z)$, for $z \in [-1, 1]$. Legendre Polynomials are $P_0(z) = 1, P_1(z) = z$, $P_2(z) = \frac{1}{2}(3z^2 - 1)$ and $P_3(z) = \frac{1}{2}(5z^3 - 3z)$. [Expand upto $P_3(z)$ term.] Legendre polynomials are orthogonal as

$$\int_{-1}^{1} P_n(z)P_m(z)dz = \frac{2}{2m+1}\delta_{mn}.$$

[Useful integral $\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$.]
[Marks distributions: Coefficients of P_0, P_1, P_2 have 3 marks and P_3 has 5 marks and 2 marks for the rest of the calculations.]

(c). Expand the same function $f(z) = \sin(az)$ via Fourier series for z real and bounded as $0 \leq z \leq \pi$ for integer values of a and for non-integer values of a. For non-integer a, express the final result in this form

$$\sin(az) = \frac{1 - \cos(a\pi)}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n \cos(a\pi)}{a^2 - n^2} \cos(nz).$$

[Useful integrals (for m & n integers)

$$\begin{aligned}
\int_0^{2\pi} \sin(mx)\sin(nx)dx &= 0 && \text{for} \quad m = 0 \\
&= \pi\ \delta_{mn} && \text{for} \quad m \neq 0 \\
\int_0^{2\pi} \cos(mx)\cos(nx)dx &= \pi\delta_{mn} && \text{for} \quad m \neq 0 \\
&= 2\pi\delta_{mn} && \text{for} \quad m = 0. \\
\int_0^{2\pi} \sin(mx)\cos(nx)dx &= 0 && \text{for all} \quad m \ \&\ n
\end{aligned}$$

]

[Marks distributions: For integer a (1 mark), and for non-integer a (9 marks).]

Q5: Solve either (a) or (b)

(a). Solve the PDE

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}(mgx\frac{\partial y}{\partial x}),$$

with the initial conditions $y(x, t = 0)$ and $y(0, t)$ are finite. (15)

[Hint: The solution of Bessel's differential equation

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + (r^2 - n^2)R(r) = 0.$$

n is integer and $R_n(r) = AJ_n(r) + BY_n(r)$ where $J_n \& Y_n$ are the Bessel's and Neumann's functions of order n. $J_0(r)$ is finite at $r = 0$ but $Y_0(r)$ is infinite. The recursion relation of the Bessel function for $n = 0$ is

$$J_n(r) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{r}{2}\right)^{n+2s}.$$

]

(b). Solve the PDE

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad c \text{ is a constant}.$$

Boundary conditions are:

-
$$y(x,t=0) = \frac{x}{a} \qquad \text{for} \qquad 0 \le x \le a.$$
$$= \frac{L-x}{L-a} \qquad \text{for} \qquad a \le x \le L.$$

  where $a > 0, \& L > a$.

-
$$y(x=0,t) = y(x=L,t) = 0 \qquad \& \qquad \left.\frac{\partial y}{\partial t}\right|_{x=0,t=0} = 0. \tag{15}$$

Q6:
Gamma functions

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad \text{for} \qquad Re(z) > 0 \tag{G1}$$
$$= 2\int_0^\infty e^{-t^2} t^{2z-1} dt \tag{G2}$$
$$= \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt \tag{G3}$$
$$= \lim_{n\to\infty} \frac{n!(z-1)!}{(z+n)!} n^z \tag{G4}$$
$$= \frac{1}{z} e^{-\gamma z} \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right)^{-1} e^{z/m} \qquad (\gamma = 0.5772156619) \tag{G5}$$

Beta functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{B1}$$
$$= 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta \; d\theta \tag{B2}$$
$$= \int_0^1 t^{p-1}(1-t)^{q-1} dt \tag{B3}$$
$$= \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt \tag{B4}$$

Zeta functions

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad Re(z) > 1 \tag{Z1}$$
$$= \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt \tag{Z2}$$

Error function

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{E1}$$
$$= 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \tag{E2}$$
$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!} \tag{E3}$$

(a). Answer either **(i)** or **(ii)** (10)

(i). Prove that $\int_0^\infty e^{-\lambda y} y^n dy = \frac{n!}{\lambda^{n+1}}$.

(ii). Prove that $\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}$.

(b). Solve any <u>two</u> (10+10)

(i). Prove $\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin(n\pi)}$ for $0 < n < 1$, using Γ, β, ζ and/or error functions.

$\Big[$ <u>Hint:</u> Use Eqs. (G5), (B1), (B4) and the series expansion of $\sin(x) = x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2\pi^2}\right)$. $\Big]$.

(ii). Prove that $\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$ for $Re(z) > 0$.

(iii). Find the asymptotic series of the error function as (for $x >> 1$)

$$erf(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n} x^{-2n}$$

where

$$\begin{aligned} n!! &= n(n-2)\dots 5.3.1 \qquad \text{for} \quad n > 0, \text{ odd} \\ &= n(n-2)\dots 6.4.2 \qquad \text{for} \quad n > 0, \text{ even} \\ &= 1 \qquad \text{for} \quad n = -1, 0 \end{aligned}$$