PH 204: Quantum Mechanics II   Problem Set 2

1. Hund's rule:

   Consider the following three possible states of two electrons placed in the $l = 1$ states in some radial quantum number n of some atom.
   (i) both electrons are in the $l = 1,\ l_z = 1$ state, with their spins forming a singlet,
   (ii) one electron is in the $l = 1,\ l_z = 0$ and the other in the $l = 1,\ l_z = 1$ state in a symmetric combination, with their spins forming a singlet,
   (iii) one electron is in the $l = 1,\ l_z = 0$ and the other in the $l = 1,\ l_z = 1$ state in an antisymmetric combination, with their spins forming a triplet.
   These three states are degenerate if the Coulomb interaction between the electrons is ignored.

   Assuming that the Coulomb interaction between the two electrons is described by the 'simplified' Hamiltonian $H_{int} = U\ \delta^3(\vec{r}_1 - \vec{r}_2)$, compute the expectation value of H_{int} in terms of U in the three states described above. Which state has the lowest
   energy if $U > 0$? (We do not need to know the explicit forms of the spatial wave functions to answer this question).

2. A hydrogen atom, which is in its ground state at time $t = -\infty$, is placed in an electric field of the form

   $$\vec{E} \quad = \quad \frac{\lambda}{\pi}\ \frac{\tau}{t^2\ +\ \tau^2}\ \hat{z}\ .$$

   Find the probability of finding the hydrogen atom in the $2P$ state (with $l_z = 0$) at time $t = \infty$. What does the probability tend to in the limit $\tau \to 0$, which corresponds to an impulse field $\vec{E} = \lambda\delta(t)\hat{z}$?

3. A particle in a one-dimensional simple harmonic potential is subjected to a force $F(t) = f cos(\gamma t)$ for a period of time $0 < t < T$, where γ is different from the frequency ω of the harmonic oscillator. If the particle is in the ground state at time $t < 0$, what is the probability of finding it in the first excited state at $t > T$ to lowest order in f ?

   What happens in the limit $\gamma \to \omega$?

4. A particle is in the ground state of the Hamiltonian of a shifted harmonic oscillator in one dimension,

   $$\begin{aligned} H \quad &= \quad H_0\ -\ \lambda x\ , \\ \text{where}\ \ H_0 \quad &= \quad \frac{p^2}{2m}\ +\ \frac{1}{2}\ m\omega^2 x^2\ , \end{aligned}$$

   at time $t < 0$. At time $t = 0$, λ is suddenly reduced to zero. What is the probability of finding the particle in the $n^{\rm th}$ eigenstate of the Hamiltonian H_0 at time $t > 0$?

5. At time $t < 0$, a particle is in the ground state of the one-dimensional potential

   $$\begin{aligned} V(x) \quad &= \quad 0 \quad \text{for} \quad 0 < x < L\ , \\ &= \quad \infty \quad \text{otherwise}\ . \end{aligned}$$

   At $t = 0$, the potential is suddenly changed to the form

   $$\begin{aligned} V(x) \quad &= \quad 0 \quad \text{for} \quad 0 < x < L'\ , \\ &= \quad \infty \quad \text{otherwise}\ , \end{aligned}$$

   where $L' > L$. Find the probabilities of finding the particle in the different eigenstates of the new Hamiltonian at $t > 0$.

   How would the answer change if the potential is changed adiabatically?

6. Consider an electron whose spin is pointing along the $\hat{z}$-axis at time $t < 0$. It is acted on by an impulse magnetic field of the form $\vec{B} = \lambda\, \delta(t)\, \hat{x}$. In which direction will the spin point at time $t > 0$?

   What is the minimum value of λ (in Tesla-sec) so that the spin points along the $-\ \hat{z}$-axis at time $t > 0$?

   Hint for the first part: Find the spin wave function at time $t > T$ if the magnetic field is given by

$$\begin{aligned} \vec{B} &= \frac{\lambda}{T}\, \hat{x} \quad \text{for} \quad 0 < t < T \ , \\ &= 0 \quad \text{for all other times.} \end{aligned}$$

   Then take the limit $T \to 0$.

7. Given an operator $\mathcal{O}$, the Heisenberg operator $\mathcal{O}(t)$ is defined as

$$\mathcal{O}(t) \;=\; e^{iHt/\hbar}\, \mathcal{O}\, e^{-iHt/\hbar} \ .$$

   For the one-dimensional simple harmonic oscillator, calculate the Heisenberg operators $x(t)$ and $p(t)$ in terms of the raising and lowering operators a and $a^\dagger$.

   Verify that the Heisenberg operators satisfy $m dx/dt = p$ and $dp/dt = -m\omega^2 x$.

8. Consider the $2S \to 1S + \gamma$ decay of the hydrogen atom.

   (a) Calculate the four differential decay rates $d\Gamma/d\Omega$ corresponding to $|2S, S_z = +\hbar/2 > \to |1S, S_z = \pm\hbar/2; R \text{ and } L >$, where $S_z = \pm\hbar/2$ denote the two possible spin states of the electron in the $1S$ state, and R (L) denote the two possible polarisations of the emitted photon.

   (The four possible decay rates of $|2S, S_z = -\hbar/2 >$ can then be obtained by similar arguments; you may just write them down by symmetry).

   (b) Now integrate the four decay rates over $d\Omega$, add them up for R and L and for $|1S, S_z = +\hbar/2 \text{ and } -\hbar/2 >$, and then average over $|2S, S_z = +\hbar/2 \text{ and } -\hbar/2 >$ with equal probability, to find the total unpolarised decay rate.

9. Suppose that two distinguishable particles with the same mass interact with each other by a central potential. Let $f(\theta)$ denote the scattering amplitude in the centre of mass frame, where θ is the angle of scattering of either one of the particles. Now, if two electrons whose spins are unpolarised interact through the same central
   potential, show that the differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} \;=\; \frac{3}{4}\, |f(\theta) - f(\pi - \theta)|^2 \;+\; \frac{1}{4}\, |f(\theta) + f(\pi - \theta)|^2 .$$

10. Rutherford experiment: Consider the scattering of an α-particle (with mass M and charge $-2e$) from an infinitely heavy atom (hence the recoil of the atom can be ignored). Assume that the atom has a point nucleus of charge $-Ze$ and an uniform spherically symmetric electron cloud of radius a surrounding it, so that the electron charge density $\rho(r)$ satisfies $(4\pi/3)a^3\rho(r) = Ze$ for $r \le a$, and $\rho(r) = 0$ for $r > a$. Use the Born approximation to calculate the differential scattering cross-section $d\sigma/d\Omega$ as a function of the scattering angle θ and the wave number k.

11. Consider scattering from a δ-function potential in three dimensions, $V(\vec{r}) = \lambda\delta^3(\vec{r})$. Use the Born approximation to compute the differential scattering cross-section $d\sigma/d\Omega$, and explain the form of its dependence on the angles θ and ϕ.

12. Consider scattering from a hard sphere, i.e., from the potential

$$\begin{aligned} V(r) &= \infty \quad \text{for} \quad r < a, \\ &= 0 \quad \text{for} \quad r > a. \end{aligned}$$

Find the phase shifts $\delta_l(k)$ in the limits $ka << l$ and $ka >> l$. Then find the total scattering cross-section in the limits $ka \to 0$ and $ka \to \infty$.

13. For the scattering potential $V(r) = \hbar^2 c/r^2$ (with $c > 0$), calculate the phase shift $\delta_l(k)$ as a function of the angular momentum l. What is the total scattering cross-section?

14. (a) Derive an expression for the Born approximation for scattering in one dimension.

Remember that there are only two scattering amplitudes in one dimension. Namely, if the initial wave function is $\psi_{initial} = e^{ikx}$, then the scattered wave function $\psi_{scattered} = \psi_{final} - \psi_{initial}$ is given by

$$\begin{aligned} \psi_{scattered} &= f_t \, e^{ikx} \quad \text{for} \quad x \to \infty \, , \\ &= f_r \, e^{-ikx} \quad \text{for} \quad x \to \, - \infty \, . \end{aligned}$$

You have to find the Born approximation for f_t and f_r.

(b) Use the Born approximation to derive expressions for f_t and f_r for scattering from a δ-function potential, $V(x) = \lambda\delta(x)$. Show that these agree with the exact expressions if λ is small.

15. A tritium atom in its ground state (denoted $^3H(1s)$) can decay into a helium ion ($^3He^+$) after emitting an electron and an antineutrino. (Assume that the last two particles play no role in this problem). This decay is a sudden process on atomic time scales. Using the wave functions of one electron in the various atomic states, calculate the probabilities that immediately after the decay, the helium ion will be found in the $1S$ and $2S$ states respectively.

16. Thomson scattering: Consider the elastic scattering of a high energy photon from an electron in an atom. (Ignore the spin of the electron in this problem). If the photon energy is much larger than the atomic binding energies (but much smaller than the rest energy of the electron), then it is sufficient to do first-order perturbation theory with the term $(e^2/2mc^2)\vec{A}^2$. (The contribution of the second-order perturbation due to $-(e/mc)\vec{A}\cdot\vec{p}$ is negligible due to the large energy denominator which appears in that calculation). Assume also that the atomic state does not change as a result of the scattering, and that the photon wavelengths are much larger than the typical size of the atomic wave function.

(a) If $\vec{\epsilon}_{\vec{k},\lambda}$ and $\vec{\epsilon}_{\vec{k}',\lambda'}$ denote the initial and final polarisation vectors, show that the differential scattering cross-section is

$$\frac{d\sigma}{d\Omega} = r_0^2 \, |\vec{\epsilon}^{\,*}_{\vec{k}',\lambda'} \cdot \vec{\epsilon}_{\vec{k},\lambda}|^2 \, ,$$

which is independent of $\vec{k}, \vec{k}'$ and the atomic state of the electron. Here, $r_0 = e^2/(mc^2)$ is called the classical radius of the electron. What is its numerical value?

(b) Compute the total unpolarised scattering cross-section by summing over the final polarisations and averaging over the initial polarisations with equal probability.

(c) Show that for 90° scattering for the photon, the final photon is linearly polarised along the direction $\vec{k} \times \vec{k}'$.

17. The process in the previous problem is called Compton scattering if the electron is free, i.e., not inside an atom. If the initial and final wavelengths of the photon are λ and λ', and the angle of scattering of the photon

is θ, use the relativistic relation between the energy and momentum of the electron, $E = \sqrt{p^2c^2 + m^2c^4}$, to show that

$$\lambda' \quad = \quad \lambda + \frac{h}{mc}\,(\,1 \,-\, \cos\theta\,)\,.$$

(Hence h/mc is called the Compton wavelength of the electron).