

Interaction Representation

So far we have discussed two ways of handling time dependence. The most familiar is the Schrödinger representation, where operators are constant, and states move in time. The Heisenberg representation does just the opposite; states are constant in time and operators move. The answer for any given physical quantity is of course independent of which is used. Here we discuss a third way of describing the time dependence, introduced by Dirac, known as the *interaction* representation, although it could equally well be called the Dirac representation. In this representation, both operators and states move in time. The interaction representation is particularly useful in problems involving time dependent external forces or potentials acting on a system. It also provides a route to the whole apparatus of quantum field theory and Feynman diagrams. This approach to field theory was pioneered by Dyson in the the 1950's. Here we will study the interaction representation in quantum mechanics, but develop the formalism in enough detail that the road to field theory can be followed.

We consider an ordinary non-relativistic system where the Hamiltonian is broken up into two parts. It is almost always true that one of these two parts is going to be handled in perturbation theory, that is order by order. Writing our Hamiltonian we have

$$H = H_0 + V \tag{1}$$

Here H_0 is the part of the Hamiltonian we have under control. It may not be particularly simple. For example it could be the full Hamiltonian of an atom or molecule. Nevertheless, we assume we know all we need to know about the eigenstates of H_0 . The V term in H is the part we do not have control over. It may be time-dependent, or difficult to handle in some other way.

In the interaction representation, operators move with H_0 . We define

$$O_I(t) = \exp(i\frac{H_0 t}{\hbar}) O_S \exp(-i\frac{H_0 t}{\hbar}), \tag{2}$$

where the subscript I means “interaction” and the subscript S means “Schrödinger .” Now just as Schrödinger and Heisenberg representations must give the same answers, we must get the same answer from the interaction representation. So we must have

$$\langle \Psi_I(t) | O_I(t) | \Psi_I(t) \rangle = \langle \Psi_S(t) | O_S | \Psi_S(t) \rangle, \tag{3}$$

where for the case where there is no explicit time dependence in H , Schrödinger states evolve as

$$|\Psi_S(t)\rangle = \exp(-i\frac{Ht}{\hbar}) |\Psi_S(0)\rangle. \tag{4}$$

Since not all the time dependence can be carried by H_0 , we must also have an interaction representation evolution operator which maps interaction states from one time to another.

$$|\Psi_I(t_2)\rangle = U_I(t_2, t_1) |\Psi_I(t_1)\rangle \tag{5}$$

Using this and the way Schrödinger states move, we can easily show for the case where H is constant in time that

$$U_I(t_2, t_1) = \exp\left(\frac{i}{\hbar}H_0t_2\right) \exp\left(-\frac{i}{\hbar}H(t_2 - t_1)\right) \exp\left(-\frac{i}{\hbar}H_0t_1\right). \quad (6)$$

The operator in the middle is just the evolution operator in the Schrödinger picture.

All this was for the case of no explicit time dependence in H . But there are many interesting problems where the Hamiltonian has explicit time dependence. In the Schrödinger picture, the operators like X and P are still constant, but things like the potential may have time dependence. For this case our Hamiltonian could be of the following form

$$H_S = H_0 + V_S(t). \quad (7)$$

It should be noted that time dependence does not change any of the basic ideas of quantum mechanics, or classical mechanics for that matter. The system still exists, and evolves by unitary transformation from one time to another. The Schrödinger equation still holds, even though energy will not in general be conserved here. So there is still an interaction picture evolution operator, but it takes a slightly different form,

$$U_I(t_2, t_1) = \exp\left(\frac{i}{\hbar}H_0t_2\right)U_S(t_2, t_1) \exp\left(-\frac{i}{\hbar}H_0t_1\right), \quad (8)$$

where U_S is the operator that evolves states in the Schrödinger picture. according to

$$|\Psi_S(t_2)\rangle = U_S(t_2, t_1)|\Psi_S(t_1)\rangle. \quad (9)$$

The Schrödinger equation still holds,

$$i\hbar \frac{\partial}{\partial t_2} |\Psi_S(t_2)\rangle = H_S(t_2) |\Psi_S(t_2)\rangle, \quad (10)$$

which implies the equation for U_S ,

$$i\hbar \frac{\partial}{\partial t_2} U_S(t_2, t_1) = H_S(t_2)U_S(t_2, t_1). \quad (11)$$

The other condition U_S must satisfy is $U_S(t_1, t_1) = 1$. We will not investigate U_S further, but proceed directly to studying U_I . Using the definition of U_I from Eq.(8) and Eq.(11), we have

$$i\hbar \frac{\partial}{\partial t_2} U_I(t_2, t_1) = \exp\left(\frac{i}{\hbar}H_0t_2\right)(-H_0 + H_S)U_S(t_2, t_1) \exp\left(\frac{-i}{\hbar}H_0t_1\right). \quad (12)$$

Using $H_S = H_0 + V_S$ this gives

$$i\hbar \frac{\partial}{\partial t_2} U_I(t_2, t_1) = \left(\exp\left(\frac{i}{\hbar}H_0t_2\right)(V_S(t_2) \exp\left(\frac{-i}{\hbar}H_0t_2\right)) \right) \left(\exp\left(\frac{i}{\hbar}H_0t_2\right)U_S(t_2, t_1) \exp\left(\frac{-i}{\hbar}H_0t_1\right) \right) \quad (13)$$

From the definition of operators in the interaction picture given in Eq.(2) we have finally

$$i\hbar \frac{\partial}{\partial t_2} U_I(t_2, t_1) = V_I(t_2) U_I(t_2, t_1), \quad (14)$$

where of course

$$V_I(t) = \exp\left(\frac{i}{\hbar} H_0 t\right) V_S(t) \exp\left(-\frac{i}{\hbar} H_0 t\right). \quad (15)$$

To see more explicitly the difference between Schrödinger and interaction picture representations of V , suppose that

$$V_S = -X_S F(t), \quad (16)$$

where we are allowing explicit time dependence in $F(t)$, but as always in the Schrödinger picture X_S is constant in time. Now going to the interaction picture, we have

$$V_I(t) = -\exp\left(\frac{i}{\hbar} H_0 t\right) X_S \exp\left(-\frac{i}{\hbar} H_0 t\right) F(t) = -X_I(t) F(t). \quad (17)$$

In the interaction picture, in addition to the explicit time dependence from $F(t)$, the X operator also moves with the Hamiltonian H_0 .

Perturbation Theory In virtually all cases where the interaction picture is used, a perturbative expansion in V_I is carried out. We expand U_I in orders, where “order n ” means that there are n powers of V_I in that term. Writing the expansion, we have

$$U_I(t_2, t_1) = \sum_{n=0}^{\infty} U_I^{(n)}(t_2, t_1). \quad (18)$$

Now one power of V_I is a term of order one, so matching orders on left and right sides of the differential equation for U_I given in Eq.(14), we obtain

$$i\hbar \frac{\partial}{\partial t_2} U_I^{(n)}(t_2, t_1) = V_I(t_2) U_I^{(n-1)}(t_2, t_1). \quad (19)$$

The 0th term is just the identity operator,

$$U_I^{(0)}(t_2, t_1) = I. \quad (20)$$

Setting $U_I^{(0)} = I$ takes care of the boundary condition that $U_I(t_1, t_1) = 1$, so all subsequent terms in the expansion must vanish when $t_1 = t_2$. This sets the lower limit of integration when we integrate Eq.(19) and we get the integral form for $U_I^{(n)}$ given in the following equation,

$$U_I^{(n)}(t_2, t_1) = -\frac{i}{\hbar} \int_{t_1}^{t_2} V_I(t') U_I^{(n-1)}(t', t_1). \quad (21)$$

To get an explicit formula for $U_I^{(n)}$, it is best to look at a few low order terms. Using $U_I^{(0)} = I$, we have

$$U_I^{(1)}(t_2, t_1) = -\frac{i}{\hbar} \int_{t_1}^{t_2} V_I(t') dt' \quad (22)$$

Turning to U_I^2 we have

$$U_I^{(2)}(t_2, t_1) = -\frac{i}{\hbar} \int_{t_1}^{t_2} V_I(t'_2) U_I^{(1)}(t'_2, t_1) dt'_2. \quad (23)$$

Putting our result for $U_I^{(1)}$ in this equation gives

$$U_I^{(2)}(t_2, t_1) = \left(-\frac{i}{\hbar}\right)^2 \int_{t_1}^{t_2} dt'_2 \int_{t_1}^{t'_2} dt'_1 V_I(t'_2) V_I(t'_1). \quad (24)$$

At this point, the pattern is clear. Later times are always further to the left. The general form for $U_I^{(n)}$ is

$$U_I^{(n)}(t_2, t_1) = \left(-\frac{i}{\hbar}\right)^n \int_{t_1}^{t_2} dt'_n \int_{t_1}^{t'_n} dt'_{n-1} \cdots \int_{t_1}^{t'_2} dt'_1 V_I(t'_n) \cdots V_I(t'_2) V_I(t'_1). \quad (25)$$

This form of the equation for $U_I^{(n)}$ can be used directly in this form. In field theory applications, it is useful to introduce techniques for writing an equivalent formula in which all time integrations start and end at the same points, in this case t_1 and t_2 .

The Forced Harmonic Oscillator A harmonic oscillator acted on by an external time dependent force is interesting for two reasons. First, it is a model for actual physical phenomena such as the quantum radiation from a known current. Second, it provides an excellent case where high order calculations can be carried out analytically in full detail. The Schrödinger representation full Hamiltonian is

$$H_S = \frac{PP}{2m} + \frac{1}{2}m\omega^2 XX - XF(t), \quad (26)$$

where

$$F(t) \rightarrow 0 \quad |t| \rightarrow \infty$$

We will take the term $-XF(t)$, as the “ V ” to be used in the interaction representation. The system will be taken to be in the oscillator ground state at $t = -\infty$, and our goal will be to obtain the interaction picture state at any subsequent time. This will be given in terms of the U_I operator as follows,

$$|\Psi_I(t)\rangle = U_I(t, -\infty)|0\rangle. \quad (27)$$

The interaction picture potential is

$$V_I(t) = -X_I(t)F(t), \quad (28)$$

where X_I moves with the oscillator Hamiltonian. In Schrödinger representation, we have

$$X_S = x_0(a + a^\dagger) \quad \text{where } x_0 = \sqrt{\frac{\hbar}{2m\omega}}. \quad (29)$$

Expressing H_0 in terms of creation and destruction operators gives

$$H_0 = \frac{PP}{2m} + \frac{1}{2}m\omega^2 XX = \hbar\omega(a^\dagger a + \frac{1}{2}). \quad (30)$$

Using H_0 to determine X_I we have

$$X_I(t) = \exp(\frac{iH_0 t}{\hbar})X_S \exp(\frac{-iH_0 t}{\hbar}) = x_0(a \exp(-i\omega t) + a^\dagger \exp(i\omega t)) \quad (31)$$

Comparing X_I and X_S we see that the interaction picture simply supplies motion at the harmonic oscillator frequency to a and a^\dagger . As usual, we can begin to see what is happening by doing some low order calculations. In first order we have

$$\begin{aligned} U_I^1(t, -\infty)|0 \rangle &= \frac{i}{\hbar} \int_{-\infty}^t dt' x_0 F(t') (a \exp(-i\omega t') + a^\dagger \exp(i\omega t')) |0 \rangle \\ &= \frac{i}{\hbar} \int_{-\infty}^t dt' x_0 F(t') \exp(i\omega t') a^\dagger |0 \rangle, \end{aligned} \quad (32)$$

where the second equality follows since $a|0 \rangle = 0$. We see that in first order the system can only reach the first excited state. Going to second order, we have

$$\begin{aligned} U_I^2(t, -\infty)|0 \rangle &= (\frac{i}{\hbar})^2 \int_{-\infty}^t dt'_2 \int_{-\infty}^{t'_2} dt'_1 x_0^2 F(t'_2) F(t'_1) \\ &\cdot (a \exp(-i\omega t'_2) + a^\dagger \exp(i\omega t'_2)) (a \exp(-i\omega t'_1) + a^\dagger \exp(i\omega t'_1)) |0 \rangle \end{aligned} \quad (33)$$

As in the first order case, the destruction operator on the far right destroys $|0 \rangle$ so only the creation operator on the far right needs to be kept. This puts the system in the first excited state. The operators on the left can then take the system to the second excited state, or back to the ground state. Let us focus on the term that ends up in the second excited state, and use the subscript “excited” for it. We have

$$(U_I^{(2)}(t, -\infty)|0 \rangle)_{excited} = (\frac{i}{\hbar})^2 x_0^2 \int_{-\infty}^t dt'_2 F(t'_2) e^{i\omega t'_2} \int_{-\infty}^{t'_2} dt'_1 F(t'_1) e^{i\omega t'_1} a^\dagger a^\dagger |0 \rangle \quad (34)$$

This term looks very similar to the first order term “squared” with one obvious difference, namely the upper limit on the t'_1 integration is at t'_2 not t . This has a remedy. We note that the integrand in Eq.(34) is symmetric under the interchange $t'_1 \leftrightarrow t'_2$. This means the integral over the region $-\infty < t'_1 < t'_2 < t$ is numerically the same as the integral over the region $-\infty < t'_2 < t'_1 < t$. It is useful to draw a simple diagram at this point, showing the regions of integration with t'_2 on the vertical axis and t'_1 on the horizontal axis. The importance of this is that if we now integrate over the region $-\infty < t'_1 < t, -\infty < t'_2 < t$, we have simply doubled the integral, which can be compensated by dividing by 2. We can then see the beginnings of a coherent state here, which recall would involve the structure $\exp(\alpha t) a^\dagger |0 \rangle$. If that is in fact what is happening here, we can see that the coherent state parameter $\alpha(t)$ must be

$$\alpha(t) = \frac{i}{\hbar} x_0 \int_{-\infty}^t F(t') e^{i\omega t'} dt'. \quad (35)$$

Using this definition, we can write

$$\begin{aligned}
 U_I^{(0)}(t, -\infty)|0\rangle + U_I^{(1)}(t, -\infty)|0\rangle + (U_I^{(2)}(t, -\infty)|0\rangle)_{excited} \\
 = \left(1 + \alpha(t)a^\dagger + \frac{1}{2}(\alpha(t)a^\dagger)^2\right)|0\rangle,
 \end{aligned} \tag{36}$$

which is the first few terms in the expansion of $\exp(\alpha(t)a^\dagger)|0\rangle$. It is possible using some more advanced operator techniques to show directly that a coherent state is indeed the result. Another route is to write a trial form assuming a coherent state, and demonstrating that this assumed form satisfies the correct differential equation. Doing that, we write

$$U_I(t, -\infty)|0\rangle = \exp(\Phi(t)) \exp(\alpha(t)a^\dagger)|0\rangle \equiv |\Psi_I(t)\rangle, \tag{37}$$

and demand that $|\Psi_I(t)\rangle$ satisfy

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = V_I(t) |\Psi_I(t)\rangle.$$

where recall that

$$V_I(t) = -x_0 F(t)(a \exp(-i\omega t) + a^\dagger \exp(i\omega t)).$$

Given that $|\Psi_I(t)\rangle$ is a coherent state with parameter $\alpha(t)$, the equation to be verified is finally

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = -x_0 F(t)(\alpha(t) \exp(-i\omega t) + a^\dagger \exp(i\omega t)) |\Psi_I(t)\rangle.$$

Determining Φ and showing that this equation is verified is part of Homework Set 9.

The forced oscillator is a case where order by order calculations can be summed up to give the complete answer. It is worth noting that an oscillator which starts in the ground state and is acted upon by a quite general force, will remain in a minimum uncertainty state for all time.