PH 204: Quantum Mechanics II   Homework 1

1. Suppose that an electron is placed in a three-dimensional harmonic potential $V(\vec{r}) \; = \; (1/2) \, m\omega^2 \, \vec{r}^2$.

   (i) What are the energies of the four lowest lying states?

   (ii) It turns out that the ground state has orbital angular momentum $l = 0$, while the next three have $l = 1$. Compute the correction to their energies due to spin-orbit coupling; this can be found exactly in this problem.

   (iii) Now let us put N electrons in the above potential. What is the ground state degeneracy of the N-particle system if $N \; = \; 3, \; 4$ and 5 ?

   (Include the spin-orbit coupling discussed above but ignore the Coulomb interaction between the electrons).

2. Use a variational wave function of the form

$$\psi(\vec{r}) \; \sim \; \exp(-\frac{\vec{r}^2}{2\Delta^2}) \; ,$$

   where Δ is a variational parameter, to estimate the ground state energy of the hydrogen atom. How does your answer compare with the exact result $E_0 = -me^4/2\hbar^2$, *i.e.*, what is the percentage error?

3. Use an appropriately chosen variational wave function (satisfying the correct boundary conditions) to estimate the ground state energy of a particle placed in the one-dimensional potential

$$\begin{aligned} V(x) \quad &= \quad \lambda x \quad \text{for} \quad x > 0 \; , \\ &= \quad \infty \quad \text{for} \quad x < 0 \; , \end{aligned}$$

   where λ is positive.

   Taking x to be the vertical coordinate, use your answer to estimate the ground state energy (in eV) of a neutron ($m \simeq 1.7 \times 10^{-27} \; Kg$) near the surface of the earth ($g \simeq 9.8 \; m/s^2$).

   [$1 \; eV \simeq 1.60 \times 10^{-19} \; Joules$, and $\hbar \simeq 1.05 \times 10^{-34} \; Joules - sec$].

4. Calculate the approximate transmission coefficient of a potential barrier given by $V(x) = A(1 - x^2) \text{for} - 1 < x < 1$ and zero otherwise. Also comment on the validity of the approximation.

5. a) Using variational principle show that the mean value of the Hamiltonian is stationary in the neighbourhood of its discrete eigen values.
   b) Consider a one dimensional attractive potential such that $V(x) < 0$ for all x. Using variational principle prove that such a potential has at least one bound state.

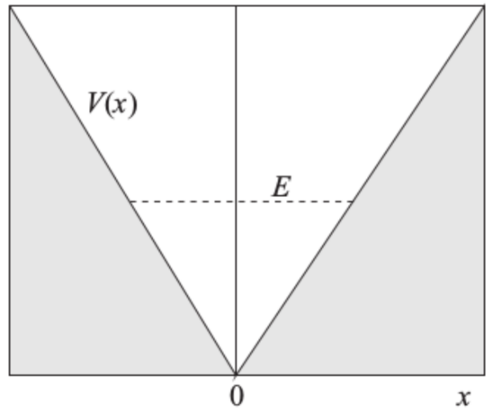
6. A particle of mass m and positive charge q moving in one dimension, is subject to a uniform electric field,

$$E(x) = E_0[\Theta(x) - \Theta(-x)]$$

   where

$$\theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$\theta(-x) = \begin{cases} 0, & x > 0 \\ 1, & x \leq 0 \end{cases}$$



(a) Take a trial wave function $\psi(x) = Ae^{-\alpha|x|}$. Normalize the wave function to obtain the constant A.
b) Use variational method to obtain the ground state energy $E_0^{Variational}$.
c) Using WKB approximation, after estimating the ground state energy E_0^{WKB}, show that $\frac{E_0^{Variational}}{E_0^{WKB}} \approx 1.49$.

7. Consider the electron moving in a Coulomb potential, $V(r) = \frac{e^2}{r}$.
a) Using the variational method,find the ground state energy for the following trial wave function : $\psi(r, \theta, \Phi, \alpha) = e^{-\alpha r^2}$.
b) Find the uncertainty in position $\Delta(X)$ for ψ and compare it with the exact result.

8. Show that

$$[\hat{L}^2, [\hat{L}^2, \hat{r}]] = 2\hbar^2(\hat{r}\hat{L}^2 + \hat{L}^2\hat{r})$$

Assuming dipole approximation, show from the above equation that no transition is possible unless $\Delta l = \pm 1$. Close the loophole by showing that if $l^{'} = l = 0$ then $< n^{'} l^{'} m^{'} \mid r \mid nlm >= 0$. Draw allowed decays for the first three Bohr levels in Hydrogen(define the levels by using the quantum numbers n,l,m). Briefly describe the transition 2S to 1S in the Hydrogen atom.

9. Calculate the energy of the second excited state of a simple harmonic oscillator choosing the trial wavefunction to be

$$\psi(x) = A(Bx^2 - 1)e^{-Cx^2}$$

Also show that this wave function satisfies all the properties of the 2nd excited state.
(Hint: Calculate E_2 as a function of C, then for some arbitrary values of m and w, numerically obtain minimum.)