PH 364 : Topological phases of matter

Marks: **70** **Mid-term** (28 Feb, 2020 at 4pm) Max time: 3 hrs

1) Symmetries

a) Determine how the following quantities transform under parity, time-reversal, charge conjugation and chiral symmetries:

(i) Electrostatic (scalar) potential. (ii) Vector potential. (iii) Hall resistivity.

(iv) Magnetic charge density (ρ_m), following the modified Maxwell's eq. $\mathbf{\nabla}.\mathbf{B} = -\rho_m$ which is invariant under all symmetries. [4x5=20]

b) Suppose we have a Hamiltonian in the 3D momentum space written in the basis of three Γ matrices as

$$H(\mathbf{k}) = \sum_{i=1}^{3} d_i(\mathbf{k})\Gamma_i$$

where the 4-component Γ matrices are defined in terms of the two sets of Pauli-matrices τ_i and σ_i matrices (i = x,y,z) as (I_2 is a 2x2 unit matrix)

$$\Gamma_1 = \tau_x \otimes I_2\,,\ \Gamma_2 = \tau_y \otimes I_2\,,\ \Gamma_3 = \tau_z \otimes \sigma_x$$

These Γ_i matrices follow the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$.

$d_i(\mathbf{k})$ are real, scalar functions of **k.**

If we set that τ_i operates on sublattice basis and σ_i operates on spin basis, this already fixes the four-component spinor, which is

$$\Psi(\mathbf{k}) = (\psi_{A\uparrow}(\mathbf{k}), \psi_{A\downarrow}(\mathbf{k}), \psi_{B\uparrow}(\mathbf{k}), \psi_{B\downarrow}(\mathbf{k}))^T.$$

Your job is to figure out if the Hamiltonian has to be invariant under a given symmetry, how $d_i(\mathbf{k})$ should transform under that particular symmetry, i.e., whether $d_i(\mathbf{k}) = \pm d_i(-\mathbf{k})$, or $d_i(\mathbf{k}) = 0$, for all *i* =1-5 components.

(i) **Parity (*P*):** Defined as

$P\psi_{A\uparrow}(\mathbf{k}) = \psi_{B\uparrow}(-\mathbf{k})$ and so on.
$PH(\mathbf{k})P^{-1} = H(-\mathbf{k})$ [5]

(ii) **Time Reversal** (*T*): Defined as

$T\psi_{A\uparrow}(\mathbf{k}) = \psi^*_{A\downarrow}(-\mathbf{k}),\ T\psi_{A\downarrow}(\mathbf{k}) = -\psi^*_{A\uparrow}(-\mathbf{k})$, and same for B
$TH(\mathbf{k})T^{-1} = H(-\mathbf{k})$ [5]

(iii) **Charge Conjugation** (C): Defined as

$$C\psi_{A\uparrow}(\mathbf{k}) = \psi^*_{A\downarrow}(\mathbf{k}), \quad C\psi_{A\downarrow}(\mathbf{k}) = \psi^*_{A\uparrow}(\mathbf{k}),$$
$$C\psi_{B\uparrow}(\mathbf{k}) = -\psi^*_{B\downarrow}(\mathbf{k}), \quad C\psi_{B\downarrow}(\mathbf{k}) = -\psi^*_{B\uparrow}(\mathbf{k}),$$
$$CH(\mathbf{k})C^{-1} = -H^T(\mathbf{k})$$ [5]

(iv) **Chiral** (S): Defined as

$$S\psi_{A\uparrow}(\mathbf{k}) = \psi_{A\uparrow}(-\mathbf{k}), \quad S\psi_{A\downarrow}(\mathbf{k}) = -\psi_{A\downarrow}(-\mathbf{k}),$$
$$S\psi_{B\uparrow}(\mathbf{k}) = -\psi_{B\uparrow}(-\mathbf{k}), \quad S\psi_{B\downarrow}(\mathbf{k}) = \psi_{B\downarrow}(-\mathbf{k})$$
$$SH(\mathbf{k})S^{-1} = -H^T(-\mathbf{k})$$ [5]

2) Ahronov-Bohm phase

The Berry phase in real space is: $\gamma_n = i\oint\langle\psi_n(\mathbf{r})|\nabla_{\mathbf{r}}\psi_n(\mathbf{r})\rangle . d\mathbf{r}$, where n is the band index.

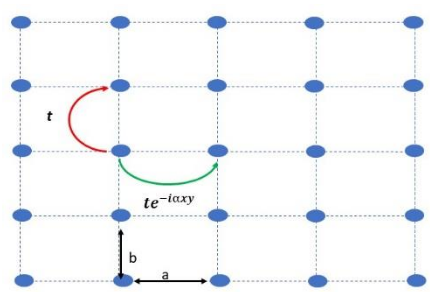
(a) Show that the Ahronov-Bohm phase (Peierls phase in a closed loop) acquired by a charge particle 'e' in a magnetic field 'B' is the same as the Berry phase acquired by the same particle.

[Hint Leibniz integral rule: $\frac{d}{db}\int_a^b f(x)dx = f(b)$] [5]

(b) Now using Laughlin's argument, and the above formula, to show that the Hall effect measures *twice* the number of electron (i.e. 2p where p is the number of electron) per Berry phase of γ (defined in unit of π). [10]

3) Quantum Hall effect without magnetic field

Let us say we are measuring the Hall effect on a square lattice without any external magnetic field. The lattice is slightly strained along the x-direction in a way that its tight-binding parameter becomes complex as $te^{-i\alpha xy}$ where α is a real constant (related to strain), and x, y are spatial variables (see figure).



Assume $\alpha = \frac{2\pi}{ab}q$, where q is a dimensionless strain parameter.

a) Estimate the effective magnetic field (called pseudo-magnetic field) for such a complex hopping parameter. [5]

b) If p number of electrons is pumped to the edge per flux of the pseudo-magnetic field through the unit cell (ab), estimate the Hall conductivity for such a system.
[Hint: Use Streda formula $\sigma_{xy} = -\frac{\partial\rho}{\partial B}$, where ρ is the charge density]. [5]

c) Plot the Hall resistivity as a function of strain parameter q.
[Hint: Electrons can be pumped in integer number.] [5]