PH205: Mathematical Methods of Physics   Total Marks: 150

Date: 27-12-2016   Quiz - 3   Maximum Time: 1.5 hrs

---

1. Solve the 2^{nd} order ODE
$$\frac{d^2\psi(r)}{dr^2} = \left(\alpha^2 + V_o \frac{e^{-\beta r}}{r}\right)\psi(r). \tag{1}$$
(α, β, V_o are real constants.)

   Note that none of the methods we have learnt in the course will be efficient to solve the entire ODE. Rather, we can solve it in the following steps.

   - First, solve the ODE with $V_o = 0$.
   - Then treat the $V_o \frac{e^{-\beta r}}{r}\psi(r)$ as the inhomogeneous or source term. (Note that usually a source term does not include $\psi(r)$, but there is no such general rule.)

   Q1. Solve the ODE
$$\frac{d^2\phi(r)}{dr^2} = \alpha^2\phi(r), \tag{2}$$
using Frobinius or series solution method. The solution can be written as
$$\begin{aligned}\phi(r) &= A\sinh(\alpha r) + B\cosh(\alpha r)\\ &= A'e^{\alpha r} + B'e^{-\alpha r},\end{aligned}$$
where $A, A'\&B, B'$ are constants to be determined.

   Hint: In the series solution, you will encounter a recursion relation between the coefficients as
$$a_{n+1} = \frac{\alpha^2 a_n}{(n+s+2)(n+s+1)}$$
where n, s are the running and constant powers, respectively as discussed in the class. (Marks: 50)

2. **Green's function:** Green's function method is an integration method to solve 2^{nd} order ODE. For an ODE with linear operator $\mathcal{L}$ acting on ψ, we have $\mathcal{L}\,\psi = f(r)$ where $f(r)$ is the source or inhomogeneous term. In this course we defined the corresponding Green's function within the range $[a, b]$ as $\mathcal{L}\,G(r, r') = \delta(r - r')$, where $a \le r' \le b$. Green's function has the two most general and useful properties that it is continuous at all values within $[a, b]$, but its first derivative is discontinuous at $r = r'$.

   Q2. Find the Green's function for the ODE in eq.(2) with the Dirichlet's boundary condition that $\phi(0) = \phi(\infty) = 0$ (Write the full Green's function in terms of both advanced and retarded Green's functions.)

   Hint:

   - The advanced Green's function ($r > r'$) is $\frac{-e^{-\alpha r}\sinh(\alpha r')}{\alpha}$.
   - The retarded Green's function ($r < r'$) is $\frac{-e^{-\alpha r'}\sinh(\alpha r)}{\alpha}$. (Marks: 50)

3. Once the Green's function for the homogeneous, linear operator is known, then the solutions for the inhomogeneous ODE can be easily evaluated using the Green's function.

   Q3a. Write the general solution $\psi(r)$ in terms of $\phi(r)$ and $G(r, r')$. (Marks: 25)

   Q3b. Then substitute the appropriate form of $\phi(r)$ and advanced, retarded Green's functions. (The final expression should be only in terms of r, r', α, β *and* V_0. The integrals are not needed to be evaluated.) (Marks: 25)