PH205:Mathematical Methods of Physics   Total Marks:150

Date:19-09-2016   Quiz 1   Maximum Time: 1 hour

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1. Gram - Schmidt procedure allows us to orthonormalize a set of functions $\{\chi_1(x),\, \chi_2(x),\, \chi_3(x),\, \ldots\}$ in a given space x to an orthonormal set $\{\varphi_1(x),\, \varphi_2(x),\, \varphi_3(x),\, \ldots\}$. In a simpler term, this procedure helps us find the unit vectors $\{\hat{e}_1,\, \hat{e}_2,\, \hat{e}_3,\, \ldots\}$ for a set of vectors $\{\overrightarrow{A_1},\, \overrightarrow{A_2},\, \overrightarrow{A_3},\, \ldots\}$ given in a coordinate system.

Step 1: We align the first unit vector $\hat{e}_1$ along any of the given vectors, say $\overrightarrow{A_1}$.

Step 2: We write down the second vector projected on the first unit vector $\hat{e}_1$ and another unit vector $\hat{e}_2$ (to be determined). Using the fact that unit vectors are orthogonal, we determine $\hat{e}_2$ from this relation. We repeat this procedure for the rest if the unit vectors.

We have studied in the class that the simple series function $\chi_n(x) = x^n$, $n = 0, 1, 2, 3, \ldots$ serves a very important basis function for generating various special polynomial functions. This is done by orthogonalizing the series with different weighting factor $w(x)$ for the inner product and with different ranges.

For example

- $w(x) = 1$, range [-1, 1] gives Legendre polynomial
- $w(x) = e^{-x}$, range $[0,\, \infty\}$ gives Laguerre polynomial
- $w(x) = (1 - x^2)^{-1/2}$, range [-1, 1] gives Chebyshev polynomial (Type I)
- $w(x) = (1 - x^2)^{1/2}$, range [-1, 1] gives Chebyshev polynomial (Type II).

Answer the followings

(a) Construct the first four Laguerre $L_n(x)$, $n = 0, 1, 2, 3$. (5+10+15+20 = 50)

(b) Do $L_n(x)$ form a vector space ? Give all the reasons. (10)

(c) Do $L_n(x)$ form a Hilbert space ? (10)

2. **Invariants:** Just as coordinate rotations leave invariant the essential properties of physical vectors, we can expect **unitary transformations** to preserve the essential features of our vector spaces. These invariants are most directly observed in the basis-set expansions of operators and functions.
Given an orthonormal basis $\{\phi_i\}$, let us define two functions $\psi = \sum b_i\phi_i$, $\chi = \sum c_i\phi_i$ where $\overrightarrow{\mathrm{b}}$ and $\overrightarrow{c}$ are vectors made of the components b_i and c_i respectively. Let A be a matrix defined in the $\{\phi_i\}$ space as $A_{ij} =< \phi_i|A|\phi_j >$.
Now let's assume another basis $\{\phi'_i\}$ related to the basis $\{\phi_i\}$ by a unitary transformation U defined as:

$$U = \begin{pmatrix} <\phi'_1|\phi_1> & <\phi'_1|\phi_2> & \cdots \\ <\phi'_2|\phi_1> & <\phi'_2|\phi_2> & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Under the unitary transformation, we set $\psi' = \sum b'_i\phi'_i$, $\chi' = \sum c'_i\phi'_i$, $A'_{ij} =< \phi'_i|A'|\phi'_j >$ and $\overrightarrow{\mathrm{b}}' = U\overrightarrow{\mathrm{b}}$, $\overrightarrow{c}' = U\overrightarrow{c}$, $A' = UAU^{-1}$.
Prove the folowing invariants:

(a) If $\overrightarrow{\mathrm{b}} = A\overrightarrow{c}$, prove that $\overrightarrow{\mathrm{b}}' = A'\overrightarrow{c}'$. The physical meaning of this identity is that all quantities must change coherently so that their relationship remains unchanged. [Note that this is one of the pre-requisites for $\overrightarrow{\mathrm{b}}$, A and $\overrightarrow{c}$ to be tensors.] (20)

(b) The inner product of ψ and χ remains invariant i.e., $< \psi|\chi >=< \psi'|\chi' >$. This is same as proving $c'^{\dagger}b' = c^{\dagger}b$. Why? (20)

(c) Expectation value of the matrix A i.e., $< A >=< \psi|A|\psi >= b^{\dagger}Ab$ remains invariant. i.e., prove that $b'^{\dagger}Ab' = b^{\dagger}Ab$ (20)

(d) Prove Trace(A) = Trace(A') (10)

(e) Prove Det(A') = Det(A) (10)